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EE 364 - Spring 2026

Week 6
Supplemental Notes

Moments

(a) Population statistics

population mean: $\mu_X = E_X[\underset{\text{r.v.}}{X}] = \begin{cases} \sum_x x \cdot f(x) & \text{discrete} \\ \int_{-\infty}^{\infty} x \cdot f(x)\,dx & \text{continuous} \end{cases}$

if sum | integral exists (i.e. finite)

$E_X[X]$ "averages out the randomness"

(b) vs. Sample statistics

sample mean: $\underset{\text{r.v.}}{\bar{X}_n} = \frac{1}{n} \sum_{k=1}^{n} \underset{\text{r.v.}}{X_k}$

(realizations: $\bar{x}_n = \frac{1}{n} \sum_{k=1}^{n} x_k$)

Defn: The population mean (expectation)

$$\underset{\text{constant}}{\mu_X} = E_X[\underset{\text{r.v.}}{X}] = \begin{cases} \sum_x x \cdot f_X(x) & X \text{ discrete} \\ \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx & X \text{ continuous} \end{cases}$$

if sum | integral exists (is finite)

Ex: Coin flip $\Omega = \{H, T\}$, $\mathcal{Q} = 2^{\Omega}$

$$X \begin{cases} \xrightarrow{p} H & +\$1 \\ \xrightarrow{1-p} T & -\$1 \end{cases}$$

reminder:

$(\Omega, \mathcal{Q}, P)$ in background.

$$E_x[X] = P[H] \cdot x_H + P[T] \cdot x_T$$
$$= p(\$1) - q(\$1)$$

$\therefore$ if $p = q \longrightarrow E_x[X] = 0$

Aside: Casino: $E_{You}[X] < 0$

Rational price $\qquad$ price $= E[\text{gain}]$

"break even" on average

vs. $\min[\text{loss}]$

Thrm: (Law of the unconscious statistician, LOTUS)

X random variable, $Y = g(X)$

$$E_Y[Y] = E_x[g(X)] = \begin{cases} \int_{-\infty}^{+\infty} g(x) \cdot f_x(x)\, dx \\ \sum_x g(x) \cdot p_x(x) \end{cases} \qquad \text{if } E[|X|] < \infty$$

Defn: The population variance

$$\sigma_x^2 = V_x[X] = E_x[(X - E_x[X])^2].$$

if finite $(\sigma_x^2 < \infty)$ $\qquad$ Note: Cauchy $\sigma_x^2 = \infty$.

Thrm: If $X \sim b(n,k,p)$ ① $E[X] = n \cdot p$ **Binomial**

② $V[X] = n \cdot p \cdot q$

Prf: ① $E[X] = \sum_{k=0}^{n} k \cdot P[X=k] = \sum_{k=0}^{n} k \cdot b(n,k,p)$

$$\overset{X \sim b}{=} \sum_{k=0}^{n} k \binom{n}{k} p^k q^{n-k} = \sum_{k=1}^{n} k \cdot \frac{n!}{k!(n-k)!} p^k q^{n-k}$$

$$= \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!} p^k q^{n-k}$$ put $j = k-1$ $\therefore k = j+1$

$$= \sum_{j+1=1}^{j+1=n} \frac{n(n-1)!}{j!(n-1-j)!} p^{j+1} q^{(n-1)-j}$$

$$= \sum_{j=0}^{n-1} \frac{n \cdot (n-1)!}{j!((n-1)-j)!} p \cdot p^j q^{(n-1)-j}$$

$$= n \cdot p \underbrace{\sum_{j=0}^{n-1} \frac{(n-1)!}{j!((n-1)-j)!} p^j q^{(n-1)-j}}_{\text{binomial theorem } = 1}$$

$$= n \cdot p.$$

② $E[X^2] = \sum_{k=0}^{n} k^2 \cdot \frac{n!}{k!(n-k)!} p^k q^{n-k}$.

$$= n \cdot p \cdot \sum_{k=1}^{n} k \cdot \frac{(n-1)!}{(k-1)!(n-k)!} p^{k-1} q^{n-k}$$ put $j = k-1$, $k = j+1$

$$= n \cdot p \cdot \sum_{j=0}^{n-1} (j+1) \frac{(n-1)!}{j!((n-1)-j)!} p^j q^{(n-1)-j}$$

$$= np \cdot \Bigg[\underbrace{\sum_{j=0}^{n-1} j \cdot \frac{(n-1)!}{j!((n-1)-j)!} p^j q^{(n-1)-j}}_{= (n-1)p \text{ by } ①} + \underbrace{\sum_{j=0}^{n-1} \frac{(n-1)!}{j!((n-1)-j)!} p^j q^{(n-1)-j}}_{=1 \text{ by BT.}} \Bigg].$$

$$= np \cdot [(n-1)p + 1] = np(np + (1-p))$$

$$= np(np+q) = (np)^2 + npq$$

$\therefore V[X] = E[X^2] - E^2[X]$

$$= \cancel{(np)^2} + npq - \cancel{(np)^2} = npq$$

QED

Proposition 1: If $Y = aX + b$ $(a \neq 0)$ and $X \sim f_x$

then. ① $E_Y[Y] = a \cdot E_x[X] + b. = a \cdot \mu_x + b$

② $V_Y[Y] = a^2 \cdot V_x[X] = a^2 \cdot \sigma_x^2$

Prf: prove for continuous, discrete replace integral / sum

① $E_Y[Y] = E_x[aX + b] \xrightarrow{\text{if: continuous}} \int_{-\infty}^{+\infty} (ax + b) f_x(x)\, dx$

$$= \int_{-\infty}^{+\infty} \left(axf_x(x) + bf_x(x) \right) dx$$

$$= \underbrace{a \int_{-\infty}^{+\infty} x \cdot f_x(x)\, dx}_{= E_x[X]} + \underbrace{b \cdot \int_{-\infty}^{\infty} f_x(x)\, dx}_{= 1}$$

$$= a \cdot E_x[X] + b$$

② $V_Y[Y] = V_x[aX + b] = E_x[(aX + b - E[aX + b])^2]$

$$= E_x[(aX + \cancel{b} - aE_x[X] - \cancel{b})^2].$$

$$= E_x[(a(X - E_x[X]))^2]$$

$$= E_x[a^2 (X - E_x[X])^2].$$

$$= a^2 \cdot \underbrace{E_x[(X - E_x[X])^2]}_{V_x[X]}$$

$$= a^2 \cdot V_x[X]$$

Proposition 2: $V_x[X] = E_x[X^2] - E_x^2[X].$

Prf: $V_x[X] = E[(X - E_x[X])^2] = E_x[X^2 - 2 \cdot X \cdot \underbrace{E_x[X]}_{\text{constant}} + (\underbrace{E_x[X]}_{\text{constant.}})^2].$

$$= E_x[X^2] - 2 \cdot \underbrace{E_x[X] \cdot E_x[X]}_{(E_x[X])^2} + (E_x[X])^2$$

$$= E_x[X^2] - (E_x[X])^2$$

Proposition 3: ("Standardization") If $Z = \frac{X-\mu}{\sigma}$ $(\sigma_X^2 < \infty)$

then: ① $E[Z] = 0$

② $V[Z] = 1$

Prf: ① $E_Z[Z] = E_X\left[\frac{X - E[X]}{\sqrt{V[X]}}\right] = E_X\left[\frac{X-\mu}{\sigma}\right]$

$= E_X\left[\frac{1}{\sigma}\cdot X - \frac{\mu}{\sigma}\right]$

$= \frac{1}{\sigma}\cdot \underbrace{E_X[X]}_{\mu.} - \frac{\mu}{\sigma}$

$= \frac{\mu}{\sigma} - \frac{\mu}{\sigma}$

$= 0.$

② $V_Z[Z] = V_X\left[\frac{1}{\sigma}X + \frac{\mu}{\sigma}\right] = \frac{1}{\sigma^2}\cdot \underbrace{V_X[X]}_{\sigma^2} = \frac{\sigma^2}{\sigma^2} = 1.$

Thrm: If $X \sim G(p)$ $\quad \mu_X = 1/p$, $\sigma_X^2 = q/p^2$ $\quad$ Geometric, $0<p<1$

$\therefore$ If $X \sim NB$: $\quad \mu_X = k/p$, $\sigma_X^2 = kq/p^2$ $\quad$ Negative binomial

since $X = X_1 + \cdots + X_k$ and independent $X_j \sim G(p)$.

Prf: ① $\mu_X = E[X] \overset{G(p)}{=} \sum_{n=1}^{\infty} n \cdot p \cdot q^{n-1} = p \sum_{n=1}^{\infty} n \cdot q^{n-1}$

$$= p \cdot \sum_{n=1}^{\infty} \frac{d}{dq} q^n \overset{R.O.C.}{=} p \cdot \frac{d}{dq}\left(\sum_{n=1}^{\infty} q^n\right)$$

since R.O.C. $p<1$

$\therefore$ uniform convergence ($\therefore$ commute)

$$= p \cdot \frac{d}{dq}\left(\frac{q}{1-q}\right) = p \cdot \frac{(1-\cancel{q}) - \cancel{q}}{(1-q)^2}$$

$$= \frac{p}{(1-q)^2} \overset{p=1-q}{=} \frac{p}{p^2} = \frac{1}{p} \quad \text{since } p>0$$

② $E[X^2] = \sum_{n=1}^{\infty} n^2 \cdot p \cdot q^{n-1} = p \cdot \sum_{n=1}^{\infty} n^2 q^{n-1}$

$$= p \cdot \sum_{n=1}^{\infty} \frac{d}{dq}(n \cdot q^n) \overset{\substack{R.O.C.\\ U.C.}}{=} p \cdot \frac{d}{dq}\left(\sum_{n=1}^{\infty} n \cdot q^n\right)$$

$$= p \cdot \frac{d}{dq}\left(q \cdot \sum_{n=1}^{\infty} n \cdot q^{n-1}\right)$$

$$= p \cdot \frac{d}{dq}\left(\frac{q}{p} \sum_{n=1}^{\infty} n \cdot p \cdot q^{n-1}\right)$$

$$= p \cdot \frac{d}{dq} \cdot \left(\frac{q}{p} \cdot E[X]\right)$$

$$\overset{①}{=} p \cdot \frac{d}{dq}\left(\frac{q}{p^2}\right) = p \cdot \frac{d}{dq}\left(\frac{q}{(1-q)^2}\right)$$

$$= p \cdot \frac{(1-q)^2 + 2q(1-q)}{(1-q)^4} \quad \text{Quotient rule}$$

$$= \frac{p}{(1-q)^2} + \frac{2pq}{(1-q)^3} = \frac{p}{p^2} + \frac{2p\cancel{q}}{p^3}$$

$p \neq 0$

$$= \frac{1}{p} + \frac{2(1-p)}{p^2} = \frac{1}{p} + \frac{2}{p^2} - \frac{2p}{p^2} = \frac{1}{p} - \frac{2}{p^2} - \frac{2}{p}$$

$$= \frac{2}{p^2} - \frac{1}{p}$$

$$\therefore \sigma_X^2 = E[X^2] - E^2[X] = \frac{2}{p^2} - \frac{1}{p} - \frac{1}{p^2}$$

$$= \frac{1}{p^2} - \frac{1}{p} = \frac{1-p}{p^2}$$

$$= \frac{q}{p^2}$$

QED.

Thrm: If $X \sim U[a, b]$ ① $E[X] = \frac{a+b}{2}$ Uniform.

② $V[X] = \frac{(b-a)^2}{12}$

Prf: ① $E_X[X] = \int_{-\infty}^{+\infty} x \underbrace{f_X(x)}_{= \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \text{else} \end{cases}} dx = \int_a^b x \frac{1}{b-a} dx$

$$= \frac{1}{b-a} \int_a^b x \, dx = \frac{1}{b-a} \left(\frac{x^2}{2} \Big|_{x=a}^{x=b} \right.$$

$$= \frac{1}{\cancel{b-a}} \cdot \frac{\overbrace{b^2 - a^2}^{(a+b)\cancel{(b-a)}}}{2}$$

$$= \frac{a+b}{2}$$

② $E_X[X^2] = \int_{-\infty}^{+\infty} x^2 \cdot f_X(x) \, dx = \int_a^b x^2 \cdot \frac{1}{b-a} dx$

$$= \frac{1}{b-a} \int_a^b x^2 \, dx = \frac{1}{b-a} \left(\frac{x^3}{3} \Big|_{x=a}^{x=b} \right.$$

$$= \frac{1}{\cancel{b-a}} \cdot \frac{\overbrace{b^3 - a^3}^{\cancel{(b-a)}(b^2+ab+a^2)}}{3}$$

$$= \frac{b^2 + ab + a^2}{3}$$

$$\therefore V_X[X] = E_X[X^2] - (E_X[X])^2$$

$$= \frac{4}{4} \cdot \frac{b^2 + ab + a^2}{3} - \left(\frac{a+b}{2} \right)^2 \cdot \frac{3}{3}$$

$$= \frac{4b^2 + 4ab + 4a^2}{12} - \frac{3a^2 + 6ab + 3b^2}{12}$$

$$= \frac{b^2 - 2ab + a^2}{12}$$

$$= \frac{(b-a)^2}{12}$$

Thrm: If $X \sim C[m, d]$ (Cauchy)

① $E[X] =$ (does not exist)

② $V[X] =$ (∞)

Prf: ① $E_x[X] = \int_{-\infty}^{\infty} x \cdot f_x(x)\, dx = \int_{-\infty}^{\infty} x \cdot \frac{1}{\pi d\left(1+\left(\frac{x-m}{d}\right)^2\right)}\, dx.$

Let $u = \frac{x-m}{d}$ $du = \frac{dx}{d}$

$$= \int_{-\infty}^{\infty} (ud+m)\frac{1}{\pi(1+u^2)}\, du$$

$$= \frac{d}{\pi} \cdot \int_{-\infty}^{+\infty} \frac{u}{1+u^2}\, du + \frac{m}{\pi}\underbrace{\int_{-\infty}^{+\infty} \frac{1}{1+u^2}\, du}_{\text{pdf}_s = 1.}$$

$$= \frac{d}{\pi}\underbrace{\int_{-\infty}^{+\infty} \frac{u}{1+u^2}\, du}\, + \frac{m}{\pi}$$

$$\int_{-\infty}^{\infty} \left|\frac{u}{1+u^2}\right| du \le \int_{-\infty}^{\infty} \left|\frac{u}{u^2}\right| du$$

$$\boxed{= \int_{-\infty}^{+\infty} \left|\frac{1}{u}\right| du = \infty}$$

② $E_x[X^2] = \int_{-\infty}^{\infty} x^2 \cdot f_x(x)\, dx = \int_{-\infty}^{+\infty} x^2 \cdot \frac{1}{1+x^2}\, dx = \int_{-\infty}^{+\infty} \frac{x^2}{1+x^2}\, dx$

$$\le \int_{-\infty}^{+\infty} \frac{x^2}{x^2}\, dx$$

$$= \int_{-\infty}^{+\infty} 1\, dx = \infty.$$

Note: Variance $\sigma_X^2 = E_X[\underbrace{(X-\mu_X)^2}_{\text{"distance"}}]$ type of dispersion.

average

absolute value

Variance $\neq$ Dispersion, e.g., could use $|\cdot|$ instead of $(\cdot)^2$

↳ but math more difficult.

Defn: The k^{th} higher order moment

$$E_X[X^k] = \int_{-\infty}^{\infty} x^k \cdot f_X(x)\,dx \qquad (\text{if} < \infty)$$

In general: k^{th} moment exists $\longrightarrow$ $(k-1)^{th}$ moment exists ($\not\longleftarrow$)

$\therefore \quad \sigma_X^2 < \infty \implies |\mu_X| < \infty$

Thm: If $k \leq m$ and $k \in \mathbb{Z}^+$ and $m \in \mathbb{Z}^+$ then

$E[X^m]$ exists $\longrightarrow$ $E[X^k]$ exists.

Pf: (continuous case, replace integrals with sums for discrete)

$$\int_{-\infty}^{\infty} |x|^k f_X(x)\,dx = \int_{|x|\leq 1} |x|^k \cdot f_X(x)\,dx + \int_{|x|>1} |x|^k \cdot f_X(x)\,dx$$

$$\leq \int_{|x|\leq 1} |x|^k \cdot f_X(x)\,dx + \int_{|x|>1} |x|^m \cdot f_X(x)\,dx$$

since $k \leq m$ and $|x|>1$

$$\leq \int_{|x|\leq 1} f_X(x)\,dx + \int_{|x|>1} |x|^m f_X(x)\,dx.$$

since $|x|\leq 1 \longrightarrow |x|^k$

$$\leq \underbrace{\int_{-\infty}^{\infty} f_X(x)\,dx}_{=1 \text{ (pdf)}} + \int_{|x|>1} |x|^m \cdot f_X(x)\,dx$$

since $\{|x|\leq 1\} \subset \mathbb{R}$

$$\leq \quad 1 + \int_{-\infty}^{\infty} |x|^m \cdot f_X(x)\, dx.$$

since $\{|x|>1\} \subset \mathbb{R}$

$$= \quad 1 + E_X\left[\,|X|^m\,\right].$$

$$< \quad \infty$$

since $E[X^m]$ exists by hypo.

$\therefore \quad E[X^k]$ exists.

QED.

Normal moments

★ **Thm:** If $Z \sim N(0,1)$ (standard normal)

then
$$E[Z^k] = \begin{cases} 0 & \text{if } k \text{ odd} \\ (k-1)(k-3)\cdots 5\cdot 3\cdot 1 & \text{if } k \text{ even} \end{cases}$$

In general: If $X \sim N(\mu, \sigma^2)$ then

$$\underbrace{E[(X-\mu_X)^k]}_{k^{th}\text{ "central moment"}} = \begin{cases} 0 & \text{if } k \text{ odd} \\ \sigma^2 (k-1)(k-3)\cdots 5\cdot 3\cdot 1 & \text{if } k \text{ even} \end{cases}$$

Prf: Case 1, k even

$$E[Z^k] = \frac{1}{\sqrt{2\pi}} \int_{z=-\infty}^{z=\infty} z^k e^{-z^2/2}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^{k-1} \left(z \cdot e^{-z^2/2}\, dz \right)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u \cdot dv$$

if $u = z^{k-1}$, $du = (k-1) z^{k-2}\, dz$; $dv = z e^{-z^2/2}\, dz$, $v = e^{-z^2/2}$

integration by parts

$$= \frac{1}{\sqrt{2\pi}} \left(uv \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} v \cdot du \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(\underbrace{\lim_{z\to\infty} z^{k-1} \cdot e^{-z^2/2} \Big|_{-z}^{z}}_{=0,\ \text{L'Hospital's}} + (k-1) \cdot \int_{-\infty}^{\infty} z^{k-2} e^{-z^2/2}\, dz \right.$$

$$= \frac{1}{\sqrt{2\pi}} (k-1) \cdot \int_{-\infty}^{\infty} z^{k-2} e^{-z^2/2} \, dz$$

$$= (k-1) \cdot E[X^{k-2}]$$

k-even

$$= (k-1)(k-3)(k-5) \cdots 5 \cdot 3 \cdot 1$$

<u>Case 2;</u> k odd

$e^{-z^2/2}$ even function

$\therefore$ $z^k e^{-z^2/2}$ odd function if k odd

$$\therefore \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^k e^{-z^2/2} \, dz = 0.$$

Ex: exponential moments $X \sim \exp(\theta)$

Thrm. $\boxed{E[X^k] = k!\, \theta^k \quad \text{if} \quad X \sim \exp(\theta)}$

Pf:

$$E[X^k] = \int_{-\infty}^{\infty} x^k \cdot f_x(x)\, dx.$$

$$= \int_0^{\infty} x^k \cdot \left(\frac{1}{\theta} \cdot e^{-x/\theta}\right) dx \qquad \frac{\theta^k}{\theta^k}$$

since $X \sim \exp(\theta)$

$$= \theta^{k-1} \int_0^{\infty} \left(\frac{x}{\theta}\right)^k e^{-x/\theta}\, d\theta$$

$\therefore$ let $u = \frac{x}{\theta}$ $\quad du = \frac{dx}{\theta}$.

$\therefore$ $dx = \theta \cdot du$

$x = 0 \longrightarrow u = 0$
$x = \infty \longrightarrow u = \infty$

$$= \theta^{k-1} \cdot \int_0^{\infty} u^k e^{-u}\, \theta \cdot du$$

$$= \theta^k \cdot \underbrace{\int_0^{\infty} u^{(k+1)-1} e^{-u}\, du}_{\Gamma(k+1)}$$

$$= \theta^k \cdot \Gamma(k+1)$$

$$= \theta^k \cdot k!$$ since $\Gamma(\alpha+1) = \alpha \cdot \Gamma(\alpha)$.

Corr: $k! = E[X^k]$ if $X \sim \exp(1)$.

Ex: If $X \sim \exp(\theta)$, then

$$\mu_X = E[X^1] = \theta$$

$$\therefore \quad \sigma_X^2 = E[X^2] - E^2[X] = 2\theta^2 - \theta^2 = \theta^2$$

More generally:

Thm: $E[X^k] = \theta^k \dfrac{\Gamma(\alpha+k)}{\Gamma(\alpha)}$ $\begin{matrix} x>0 \\ \alpha>0 \\ \theta>0 \end{matrix}$ if $X \sim \gamma(\alpha, \theta)$

Prf: $E[X^k] = \displaystyle\int_{-\infty}^{\infty} x^k \cdot f_X(x)\, dx$

$$= \frac{1}{\Gamma(\alpha)\theta^\alpha} \cdot \int_0^\infty x^k \cdot x^{\alpha-1} e^{-x/\theta} dx$$

$$= \frac{1}{\Gamma(\alpha)\theta^\alpha} \int_0^\infty x^{(\alpha+k)-1} e^{-x/\theta} dx.$$

Let $x = u \cdot \theta \qquad dx = \theta \cdot du.$

$$= \frac{1}{\Gamma(\alpha) \cdot \theta^\alpha} \int_0^\infty (u \cdot \theta)^{(\alpha+k)-1} e^{-u}\, \theta\, du$$

$$= \frac{\theta^{(\cancel{\alpha}+k)}}{\Gamma(\alpha) \cdot \cancel{\theta^\alpha}} \cdot \underbrace{\int_0^\infty u^{(\alpha+k)-1} e^{-u} du.}_{\Gamma(\alpha+k)}$$

$$= \theta^k \cdot \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}$$

QED.

Checking $\alpha = 1$: $\Gamma(\alpha+k) = \Gamma(k+1) = k \cdot \Gamma(k) = k!$

$\therefore E[X^k] = \theta^k\, k!$ if $X \sim \mathrm{Exp}(\theta)$

Corr: ① $\mu_x = \alpha\theta$

② $\sigma_x^2 = \alpha\theta^2$

if $X \sim \Gamma(\alpha, \theta)$

Pf: $\mu_x = E[X^1] \overset{X\sim\gamma}{=} \theta^1 \cdot \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)}$

$= \theta \cdot \frac{\alpha \cdot \cancel{\Gamma(\alpha)}}{\cancel{\Gamma(\alpha)}}$

$= \alpha\,\theta.$

$\sigma_x^2 = E[X^2] - E^2[X] = \theta^2 \cdot \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)} - \alpha^2 \cdot \theta^2.$

$= \theta^2 \left[\frac{(\alpha+1)\Gamma(\alpha+1)}{\Gamma(\alpha)} - \alpha^2 \right].$

$= \theta^2 \cdot \left[\frac{(\alpha+1)\alpha \cdot \cancel{\Gamma(\alpha)}}{\cancel{\Gamma(\alpha)}} - \alpha^2 \right].$

$= \theta^2 \left[\alpha^2 + \alpha - \alpha^2 \right].$

$= \alpha \cdot \theta^2.$

QED

Ex: $X \sim \chi^2(r) \rightarrow \gamma(\alpha, \theta)$, $\alpha = \frac{r}{2}$, $\theta = 2$

$\therefore E[X^1] = \theta^k \cdot \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} = 2^1 \cdot \frac{\overbrace{\Gamma\left(\frac{r}{2}+1\right)}^{\frac{r}{2}\cdot\cancel{\Gamma\left(\frac{r}{2}\right)}}}{\cancel{\Gamma\left(\frac{r}{2}\right)}} = 2 \cdot \frac{r}{2} = r.$

$E[X^2] = \theta^k \cdot \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} = 2^2 \cdot \frac{\overbrace{\Gamma\left(\frac{r}{2}+2\right)}^{\left(\frac{r}{2}+1\right)\left(\frac{r}{2}\right)\cancel{\Gamma\left(\frac{r}{2}\right)}}}{\cancel{\Gamma\left(\frac{r}{2}\right)}} = r \cdot (r+2)$

$= r^2 + 2r$

$\therefore V[X] = E[X^2] - (E[X])^2 = \cancel{r^2} + 2r - \cancel{(r)^2} = 2r.$

$= \alpha\theta^2 = \frac{r}{2} \cdot 2^2 = 2r$

Defn: Beta function $B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx$ $\quad \alpha>0,\ \beta>0$

$$= \frac{\Gamma(\alpha)\cdot\Gamma(\beta)}{\Gamma(\alpha+\beta)} \qquad \text{(see below)}$$

(optional)

Thrm: $E[X^k] = \dfrac{B(\alpha+k,\beta)}{B(\alpha,\beta)}$ if $X \sim Beta(\alpha,\beta)$ and $k>0$

Pf:
$$E[X^k] = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\Gamma(\beta)} \underbrace{\int_0^1 x^{(\alpha+k)-1}(1-x)^{\beta-1}\,dx}_{B(\alpha+k,\beta)}$$

$$= \frac{B(\alpha+k,\beta)}{B(\alpha,\beta)}$$

Corr: $E[X^k] = \dfrac{\alpha+k-1}{\alpha+\beta+k-1}\cdot E[X^{k-1}]$ $\quad (k \geq 1).$

Pf:
$$E[X^k] \overset{thrm}{=} \frac{B(\alpha+k,\beta)}{B(\alpha,\beta)}$$

$$= \frac{\Gamma(\alpha+k)\cdot\Gamma(\beta)}{\Gamma(\alpha+\beta+k)}\cdot\frac{1}{B(\alpha,\beta)}$$

$$= \frac{(\alpha+k-1)\cdot\Gamma(\alpha+k-1)\cdot\Gamma(\beta)}{(\alpha+\beta+k-1)\Gamma(\alpha+\beta+k-1)}\cdot\frac{1}{B(\alpha,\beta)}$$

$$= \frac{\alpha+k-1}{\alpha+\beta+k-1}\cdot\frac{B(\alpha+k-1,\beta)}{B(\alpha,\beta)}$$

$$= \frac{\alpha+k-1}{\alpha+\beta+k-1}\cdot E[X^{k-1}]$$

QED.

Corr. If $X \sim Beta(\alpha, \beta)$

① $\mu_X = \dfrac{\alpha}{\alpha+\beta}.$

② $\sigma_X^2 = \dfrac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$

Pf:

$$\mu_X = E[X^1] = \frac{B(\alpha+1, \beta)}{B(\alpha, \beta)}$$

$$= \frac{\Gamma(\alpha+1)\,\Gamma(\beta)}{\Gamma(\alpha+\beta+1)} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)}$$

$$= \frac{\alpha \cdot \cancel{\Gamma(\alpha)}\,\cancel{\Gamma(\beta)}}{(\alpha+\beta)\cdot \cancel{\Gamma(\alpha+\beta)}} \cdot \frac{\Gamma(\cancel{\alpha+\beta})}{\cancel{\Gamma(\alpha)}\,\cancel{\Gamma(\beta)}}$$

$$= \frac{\alpha}{\alpha+\beta}.$$

$$E[X^2] = \frac{B(\alpha+2, \beta)}{B(\alpha, \beta)} = \frac{\Gamma(\alpha+2)\cdot\cancel{\Gamma(\beta)}}{\Gamma(\alpha+\beta+2)} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\cancel{\Gamma(\beta)}}$$

$$= \frac{(\alpha+1)\cdot\Gamma(\alpha+1)}{(\alpha+\beta+1)\cdot\Gamma(\alpha+\beta+1)} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}$$

$$= \frac{(\alpha+1)\,\alpha\cdot\cancel{\Gamma(\alpha)}}{(\alpha+\beta+1)(\alpha+\beta)\cdot\cancel{\Gamma(\alpha+\beta)}} \cdot \frac{\cancel{\Gamma(\alpha+\beta)}}{\cancel{\Gamma(\alpha)}}$$

$$= \frac{\alpha(\alpha+1)}{(\alpha+\beta+1)(\alpha+\beta)}.$$

$$\therefore \sigma_X^2 = E[X^2] - E^2[X] = \frac{\alpha(\alpha+1)}{(\alpha+\beta+1)(\alpha+\beta)} - \frac{\alpha^2}{(\alpha+\beta)^2}$$

$$= \frac{\alpha(\alpha-1)(\alpha+\beta) - \alpha^2(\alpha+\beta+1)}{(\alpha+\beta+1)(\alpha+\beta)^2} = \frac{\cancel{\alpha^3} + \cancel{\alpha^2\beta} + \cancel{\alpha^2} + \alpha\beta - \cancel{\alpha^3} - \cancel{\alpha^2\beta} - \cancel{\alpha^2}}{(\alpha+\beta+1)(\alpha+\beta)^2}$$

$$= \frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$$

QED

(optional)

Thrm: $B(\alpha,\beta) = \int_0^1 u^{\alpha-1}(1-u)^{\beta-1}\,du$ $\quad \alpha>0,\ \beta>0$

$$= \frac{\Gamma(\alpha)\cdot\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

Prf: $\Gamma(\alpha)\cdot\Gamma(\beta) = \int_{x=0}^{x=\infty} e^{-x}x^{\alpha-1}\,dx \int_{y=0}^{y=\infty} e^{-y}y^{\beta-1}\,dy$

$$\overset{\text{Fubini}}{=} \int_{y=0}^{y=\infty}\left[\int_{x=0}^{x=\infty} e^{-(x+y)}x^{\alpha-1}y^{\beta-1}\,dx\right]dy$$

$$= \int_{y=0}^{y=\infty}\int_{x=0}^{x=\infty} f\big(x(u,v),\,y(u,v)\big)\,dx\,dy$$

double substitution

Let: $x = x(u,v) = uv \qquad y = y(u,v) = u(1-v)$

Given: $0<x<\infty$ and $0<y<\infty$

$\therefore\ 0 < x+y = u\cdot v + u(1-v) = \cancel{uv} + u - \cancel{uv} = u$

$\therefore\ u>0$

$\therefore\ x>0 \longrightarrow uv>0$

$\therefore\ v>0$ since $u>0$

$\therefore\ u<\infty$ since $x = u\cdot v<\infty$ and $v>0$

$\therefore\ 0<u<\infty$ $\longleftarrow$ first limit of integration

$\therefore\ y>0 \longrightarrow u(1-v)>0$ $\qquad \therefore\ 1-v>0$ (since $u>0$)

$\therefore\ v<1$ $\qquad \therefore\ 0<v<1$ $\longleftarrow$ second limit of integration

$$\therefore \Gamma(\alpha)\cdot\Gamma(\beta) \overset{\text{CVT}}{=} \int_{v=0}^{v=1}\int_{u=0}^{u=\infty} f(u,v)\cdot\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

(change of variable theorem)

$$= \int_{v=0}^{v=1}\int_{u=0}^{u=\infty} f(u,v)\; u\cdot du\,dv.$$

since $\underbrace{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|}_{\text{absolute determinant of Jacobian matrix}} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix}$ since $x = u\cdot v$. $y = u - uv$

$$= \left| -\cancel{vu} - u + \cancel{vu} \right| = |-u| = u.$$

$$= \int_{v=0}^{v=1}\int_{u=0}^{u=\infty} e^{-u}(uv)^{\alpha-1}\left(u(1-v)\right)^{\beta-1} u \;du\,dv$$

$$= \int_{v=0}^{v=1}\int_{u=0}^{u=\infty} e^{-u}\, u^{\alpha-1} v^{\alpha-1} u^{\beta-1}(1-v)^{\beta-1} u^{1}\, du\,dv$$

$$= \left[\int_{u=0}^{u=\infty} e^{-u} u^{\alpha-1+\beta-1+1}\,du\right]\cdot\left[\int_{v=0}^{v=1} v^{\alpha-1}(1-v)^{\beta-1}\,dv\right]$$

$$= \left[\int_{u=0}^{u=\infty} e^{-u} u^{(\alpha+\beta)-1}\,du\right]\cdot\left[\int_{v=0}^{v=1} v^{\alpha-1}(1-v)^{\beta-1}\,dv\right]$$

$$= \Gamma(\alpha+\beta)\cdot B(\alpha,\beta)$$

$$\therefore B(\alpha,\beta) = \frac{\Gamma(\alpha)\cdot\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

QED

Thrm: ① $E[I_A] = \overset{p}{P(A)}$

② $V[I_A] = P(\overset{p}{A}) \cdot P(\overset{q}{A^c})$

for any $A \in \mathcal{A}$ in probability space $(\Omega, \mathcal{A}, P)$

Prf: ① $E[I_A] = \int_\Omega I_A(w)\, dP$

$$= \int_A \underbrace{I_A(w)}_{=1\ w \in A}\, dP + \int_{A^c} \underbrace{I_{A^c}(w)}_{=0\ w \notin A}\, dP \qquad \text{since } \Omega = A \cup A^c$$

$$= \int_A 1 \cdot dP + \underbrace{\cancel{\int_{A^c} 0 \cdot dP}}_{=0} = \int_A dP$$

$$= P(A)$$

② $V[I_A] = \int_\Omega (I_A(w) - E[I_A])^2\, dP$

$$= \int_\Omega (I_A(w) - P(A))^2\, dP$$

$$= \int_\Omega (I_A^2(w) + P^2(A) - 2 \cdot P(A) \cdot I_A(w))\, dP$$

$$= \int_\Omega I_A^2(w)\, dP + P^2(A) \int_\Omega dP - 2 \cdot P(A) \cdot \int_\Omega I_A(w)\, dP$$

$$= \int_\Omega I_A(w)\, dP + P^2(A) - 2P^2(A)$$

since $I_A^2 = I_A$ for binary sets (not fuzzy)

$$= P[A] - P^2[A] = P(A) \cdot (1 - P(A))$$

$$= P(A) \cdot P(A^c)$$

QED